MATH2060B Appendix: Riemann-Lebesgue Lemma

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1 Introduction

The Riemman-Lebesgue Lemma is an important and fundamental result in the study of Fourier analysis. The proof of the Lemma can be found in many standard real analysis text books (see for example, [1]) for the case of Lebesgue integrable functions which is the generalization of Riemann integrable functions. It is worthwhile mentioning that one can employ some basic knowledge in functional analysis to obtain a simple proof of this result (see [2]). In this note, we will prove the Lemma for the case of Riemann integrable functions.
Let us first recall the Riemann-Lebesgue Lemma.

Theorem 1.1 *(Riemman-Lebesgue Lemma) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Then we have*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \; dx = 0.$$

2 The proof

Recall that a function φ on a closed and bounded interval $[a, b]$ is called a step function if there is a partition $a = x_0 < \cdots < x_l = b$ such that φ is a constant c_i on each (x_{i-1}, x_i). In this case, $\varphi \in R[a, b]$ and

$$\int_a^b \varphi(x) \; dx = \sum_{i=1}^{l} c_i(x_i - x_{i-1}).$$

Before showing the main result, we need the following lemma first.

Lemma 2.1 *Let $f \in R[a, b]$. Then for any $\varepsilon > 0$, there is a step function φ on $[a, b]$ such that $\varphi \leq f$ on $[a, b]$ and*

$$|\int_a^b f - \int_a^b \varphi| < \varepsilon.$$

Proof: Let $\varepsilon > 0$. By using the definition of Riemann integrable function, there is a partition P on $[a, b]$ such that

$$|\int_a^b f(x)dx - L(f, P)| < \varepsilon,$$

where $L(f, P)$ denotes the lower sum of f with respect to the partition $P : a = x_0 < \cdots < x_l = b$ over $[a, b]$. So, if we put $\varphi(x) = \inf_{t\in[x_{i-1},x_i]} f(t)$ for $x \in (x_{i-1}, x_i)$ and $\varphi(x_i) := f(x_i)$ for all $i = 1, 2, , ...$ then the step function φ is as desired. □

We are now to give the proof of Riemann-Lebesgue Lemma for the case of Riemann integrable functions over $[a, b]$.

Theorem 2.2 *Let f be a Riemann integrable function over $[a, b]$. Then we have*

$$\lim_{n\to\infty} \int_a^b f(x)\cos nx \ dx = 0.$$

Proof: Let $\varepsilon > 0$. Then by Lemma 2.1, one can find a step function φ on $[a, b]$ with $\varphi \le f$ such that $|\int_a^b f - \int_a^b \varphi| < \varepsilon$. This implies that

$$|\int_a^b f(x)\cos nx \ dx - \int_a^b \varphi(x) \ \cos nx dx| \le \int_a^b (f(x) - \varphi(x))|\cos nx| \ dx < \varepsilon$$

for all $n = 0, 1, 2.....$ Thus, it suffices to show that $\lim_{n\to\infty} \int_a^b \varphi(x)\cos nx \ dx = 0$. Now for each non-empty subset A of $\mathbb{R}$, put $\chi_A(x) \equiv 1$ for $x \in A$; otherwise, is 0. If we write $\varphi(x) = \sum_{i=1}^{l} c_i\chi_{[x_{i-1},x_i)}(x)$, where $a = x_0 < \cdots < x_l = b$ and c_i's are constants, then we have

$$\begin{aligned}
|\int_a^b \varphi(x)\cos nx \ dx| &= |\sum_{i=1}^{l} \int_{x_{i-1}}^{x_i} c_i \cos nx \ dx| \\
&\le \sum_{i=1}^{l} \frac{|c_i|}{n}|(\sin nx_i - \sin nx_{i-1})| \\
&\le 2\sum_{i=1}^{l} \frac{|c_i|}{n} \to 0
\end{aligned}$$

as $n \to \infty$. The proof is finished. □

Theorem 2.3 *If f is absolutely Riemann integrable over $\mathbb{R}$, i.e., $\int_{-\infty}^{\infty} |f(x)| \ dx < \infty$, then we have*

$$\lim_{n\to\infty} \int_a^b f(x)\cos nx \ dx = 0.$$

Proof: Let $\varepsilon > 0$. Using the Cauchy criterion, we see that the function $f(x)\cos nx$ is also absolutely Riemann integrable over $\mathbb{R}$, moreover, there is $M > 0$ so that

$$\int_{-\infty}^{-M} |f(x)||\cos nx| \ dx + \int_M^{\infty} |f(x)||\cos nx| \ dx < \varepsilon$$

for all $n = 0, 1, 2, ...$ Applying Theorem 2.2 for the restriction of f on $[-M, M]$, then there is a positive integer N so that

$$\int_{-M}^{M} |f(x)||\cos nx| \ dx < \varepsilon$$

for all $n \ge N$. The proof is finished. □

Remark 2.4 Note that the Lebesgue integrability is equivalent to the absolutely convergence of integrals. Therefore, we don't need to assume $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ in Theorem 2.1. However, this is not the case for the class of Riemann integrable functions over $\mathbb{R}$.

References

[1] H. Royden and P. Fitzpatrick, Real analysis, $4th$-edition, Pearson, (2010).

[2] E.M. Stein and R. Shakarchi, Fourier analysis, Princeton, (2003).